THE SECULAR DISPLACEMENT OF THE ORBIT OF A SATELLITE.

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(1). IN his Mécanique Céleste, Tome II, p. 373, Laplace has considered the cause that holds the Rings of Saturn nearly in the plane of the equator of the planet, and he has explained the reason that led him to predict the rotation of Saturn and its Rings before Herschel had determined these rotations by observation. In this investigation Laplace employs the equations that determine the rotation of a solid body, the disturbing forces being those that arise from the action of the excess of matter around the equator of the planet and the attraction of a distant body. In the same chapter it is also shown why a satellite will be held nearly in the equator of its primary when the figure of the planet is elliptical. These results have been used by Professor J. C. Adams to explain why the orbits of the satellites of Mars are at the present time nearly coincident with its equator. M. F. Tisserand has discussed the same question by means of formulæ derived by him in a discussion of the secular perturbations of the orbit of Japetus, the outer satellite of Saturn. The investigations of these astronomers render it probable that the figure of Mars is slightly elliptical, and that the orbits of its satellites will always remain nearly coincident with the equator of the planet.

The question of the perturbations of satellites has been treated by Laplace in the fourth volume of his Me'canique Ce'leste. There Laplace determines the perturbations of the radius vector, and of the longitude and latitude of the satellite; and for astronomical purposes these coordinates seem to be as good as any that can be chosen. If we wish to know, however, the effect of the perturbation on any element of the orbit, it is more convenient to use the formulæ of Lagrange. In the unfinished essay on the Saturnian System left by Bessel he has given these formulæ, and also the various forms of the perturbative, or potential function that come into use in this complicated and interesting system. M. C. Souillart has lately published an elaborate memoir on the satellites of Jupiter in which he verifies and extends the investigations of Laplace referred to above.

The forces that disturb the motion of a satellite are, (1), the action of the protuberant matter around the equator of the primary planet; (2), the attraction of another satellite of this planet; (3), the attraction of a body outside this system, like the sun or another planet; and in the case of Saturn

the action of its Ring also produces a disturbance in the motion of its satellites. Since the perturbations are small they may be computed separately and then added together; I consider here only those which are produced by the protuberant matter around the equator of the planet, and by the action of the sun.

Denote by θ the longitude of the ascending node of the orbit of the satellite on the orbit of its primary planet, and by i the inclination of the satellite's orbit to the same plane. Let g be the angle between the line of apsides of the orbit of the satellite and its line of nodes; then if a be the semi-major axis of this orbit, e the eccentricity, and if n be the mean sidereal motion of the satellite, we shall have by the formulæ of Lagrange;

$$\begin{aligned} \frac{di}{dt} &= \frac{an}{\sin i\sqrt{(1-e^2)}} \cdot \left(\frac{dR}{d\theta}\right) - \frac{an\cos i}{\sin i\sqrt{(1-e^2)}} \cdot \left(\frac{dR}{dg}\right) \\ \frac{d\theta}{dt} &= -\frac{an}{\sin i\sqrt{(1-e^2)}} \cdot \left(\frac{dR}{di}\right). \end{aligned} \qquad (1)$$

R denotes the perturbative function used by Laplace.

We have now to expand this function for the two disturbing forces in elements of the orbit of the satellite, substitute the partial differential coefficients of R in equations (1), and then integrate them.

Assuming the satellite to be a material particle whose mass may be neglected, let x, y, z, be its rectangular coordinates referred to the centre of gravity of the planet, which is supposed to be fixed, and let

$$r = \sqrt{(x^2 + y^2 + z^2)}.$$

Let X, Y, Z, be the coordinates of the centre of the sun, S its mass, and also let

$$D = \sqrt{(X^2 + Y^2 + Z^2)}.$$

If m be the mass of the primary planet and $\frac{m}{r} + V$ the sum of the particles of the planet divided respectively by their distances from the satellite, then the value of the perturbative function will be,

$$R = \frac{S(Xx+Yy+Zz)}{D^3} - \frac{S}{[(X-x)^2+(Y-y)^2+(Z-z)^2]^{\frac{1}{2}}} - V.$$

The first two terms give the perturbations produced by the sun, and the last term those caused by the non spherical figure of the primary. We have now to develop this value of R and find those terms that produce the secular part of the perturbations. The denominator of the second term is

$$D \cdot \left\{1+\frac{r^2}{D^2}-\frac{2(Xx+Yy+Zz)}{D^2}\right\}^{\frac{1}{2}},$$

and expanding the reciprocal of this we get for the second term of R,

$$-\frac{S}{D}+\frac{S.r^2}{2D^3}-\frac{S.(Xx+Yy+Zz)}{D^3}-\frac{3S.r^4}{8D^5}$$
$$-\frac{3S.(Xx+Yy+Zz)^2}{2D^5}+\frac{3S.r^2(Xx+Yy+Zz)}{2D^5},\ \&c.$$

The first term of this expansion may be omitted since it does not contain the elements of the orbit of the satellite and will disappear in the differentiation, and the third term cancels the first term in R. Again D is very great with respect to r, and hence the approximate value of the perturbative function that depends on the action of the sun becomes

$$R_1=\frac{S.r^2}{2D^3}-\frac{3S.(Xx+Yy+Zz)^2}{2D^5}. \tag{2}$$

If a great circle be drawn from the position of the satellite in its orbit to the sun, and we call the arc joining these bodies f, we shall have,

$$\cos f=\frac{Xx+Yy+Zz}{rD},$$

and therefore

$$\frac{r^2}{D^3}\cos f^2=\frac{(Xx+Yy+Zz)^2}{D^5}.$$

If we denote by u and U the angular distances of the satellite and the sun from the node, the spherical triangle between the node, the satellite and the sun gives

$$\cos f = \cos u\cos U+\sin u\sin U\cos i,$$

and hence

$$\cos f^2 = \cos u^2\cos U^2+\sin u^2\sin U^2\cos i^2$$
$$+2\cos u\sin u\cos U\sin U\cos i.$$

Changing the powers into cosines of the multiple arcs we have

$$\cos f^2 = \tfrac{1}{4}.[2-\sin i^2+2\cos 2u\cos 2U+2\sin 2u\sin 2U\cos i$$
$$+(\cos 2u+\cos 2U-\cos 2u\cos 2U)\sin i^2].$$

As we wish to get only the secular part of this expression we have by omitting the periodical terms depending on the position of the sun,

$$\frac{r^2}{D^3}.\cos f^2 = \frac{1}{4D^3}.\left\{(2-\sin i^2).r^2+r^2\cos 2u.\sin i^2\right\}.$$

If now Φ be a function of the radius vector and true anomaly, r and v, in an ellipse, and if we denote by M and E the mean and excentric anomalies, then the non-periodical part of Φ is given by the definite integral,

$$\frac{1}{2\pi}\int_0^{2\pi}\Phi.dM=\frac{1}{2\pi\sqrt{(1-e^2)}}.\int_0^{2\pi}\Phi.\frac{r^2}{a^2}.dv=\frac{1}{2\pi}\int_0^{2\pi}\Phi.\frac{r}{a}.dE.$$

Since from the equation of the ellipse we have

$$r=a(1-e\cos E),$$

the last integral gives for the non periodical part of r^2,

$$\frac{a^2}{2\pi}\int_0^{2\pi}(1-e\cos E)^3.dE = a^2.(1+\tfrac{3}{2}e^2).$$

Again since $u = v+g$ we have to find the non periodical parts of

$$\Phi = r^2\cos 2v: \qquad \text{and} \quad \Phi = r^2\sin 2v.$$

The equation of the ellipse gives

$$\sin v = \frac{\sin E.\sqrt{(1-e^2)}}{1-e\cos E}: \qquad \cos v = \frac{\cos E-e}{1-e\cos E},$$

and hence

$$r^2\cos 2v = a^2.[(\cos E-e)^2-(1-e^2).\sin E^2],$$
$$r^2\sin 2v = 2a^2.\sin E.(\cos E-e).\sqrt{(1-e^2)}.$$

Substituting these values of Φ and integrating we have

$$\Phi = \tfrac{5}{2}a^2e^2: \qquad \text{and} \quad \Phi = 0.$$

If a_0 and e_0 be the semi-major axis and the excentricity of the orbit of the planet we shall have for the non periodical part of $\frac{1}{D^3}$,

$$\Phi = \frac{1}{2\pi a_0^3}.\int_0^{2\pi}\frac{dE}{(1-e_0\cos E)^2} = \frac{1}{a_0^3(1-e_0^2)^{\frac{3}{2}}}.$$

From these results we find for the secular part of R_1, the value

$$R_1 = \frac{S.a^2}{a_0^3(1-e_0^2)^{\frac{3}{2}}}.\left\{\left(-\frac{1}{4}+\frac{3}{8}\sin i^2\right).\left(1+\frac{3e^2}{2}\right)-\frac{15}{16}.e^2\sin i^2\cos 2g\right\}. \quad (3)$$

The term $\cos\frac{2u}{D^3}$ in $\cos f^2$ produces no constant terms of an order lower than e_0^4, and therefore it may be neglected.

(2). In the case of an ellipsoid of revolution Laplace gives a very simple form to the potential V of the disturbing force. If we denote by ρ the ellipticity of the spheroid, by φ the ratio of centrifugal force to gravity at the equator of the planet, by B the radius of this equator, and by μ the sine of the declination of the satellite with respect to the same equator, we shall have; Mec. Cel., Tome II, p. 103,

$$V = \frac{mB^2}{r^3}.\left(\frac{\varphi}{2}-\rho\right).\left(\mu^2-\frac{1}{3}\right).$$

Taking the mass of the planet and its equatorial radius for the units of mass and distance, and putting $Q' = \frac{1}{2}\varphi-\rho$, we have

$$V = \frac{Q'}{r^3}.(\mu^2-\tfrac{1}{3}).$$

Let ψ be the distance from the node of the orbit of the satellite to its intersection with the equator of the planet, and let γ be the inclination of its

orbit to this equator, then from the right angled spherical triangle we have

$$\mu = \sin\gamma \sin(u - \psi)$$

and

$$\mu^2 = \tfrac{1}{2}\sin\gamma^2 . [1 - \cos 2(u-\psi)].$$

The part of V which gives the secular terms is therefore

$$R_2 = -\frac{Q'}{2a^3(1-e^2)^{\frac{3}{2}}} . \sin\gamma^2. \tag{4}$$

Let A be the inclination of the equator of the planet to the orbit of the planet, and let θ_1 be the longitude of the node of the equator on this orbit; the spherical triangle formed by the three nodes gives,

$$\begin{aligned}\cos\gamma &= \cos(\theta_1-\theta)\sin A \sin i + \cos A \cos i\\ \sin\gamma\sin\psi &= \sin(\theta_1-\theta)\sin A\\ \sin\gamma\cos\psi &= \cos(\theta_1-\theta)\sin A\cos i - \cos A \sin i.\end{aligned}$$

These equations give the means of expressing $\sin\gamma^2$ in terms of θ and i, and hence R_2 becomes a function of the elements of the satellite's orbit. The values of A and θ_1, which determine the position of the equator of the planet on the plane of its orbit, are in fact subject to small variations similar to precession and nutation, but here these quantities are supposed to be constant.

The excentricities of the orbits of the satellites in our solar system are generally very small, and if we neglect the squares of these excentricities, and omit the constant part in R_1, equations (1), (3) and (4) give, if we put

$$\begin{aligned}Q &= \frac{3S.a^2}{8a_0^3(1-e_0^2)^{\frac{3}{2}}}\\ \frac{di}{dt} &= \frac{an}{\sin i} . \left(\frac{dR}{d\theta}\right)\\ \frac{d\theta}{dt} &= -\frac{an}{\sin i} . \left(\frac{dR}{di}\right)\\ R &= Q . \sin i^2 - \frac{Q'}{2a^3} . \sin\gamma^2;\end{aligned}$$

or we may write

$$R = \tfrac{1}{2}c . \sin i^2 + \tfrac{1}{2}c' . \sin\gamma^2.$$

Hence we find

$$\begin{aligned}\left(\frac{dR}{d\theta}\right) &= - c' . \cos\gamma . \left(\frac{d.\cos\gamma}{d\theta}\right)\\ \left(\frac{dR}{di}\right) &= c.\sin i \cos i - c'.\cos\gamma . \left(\frac{d.\cos\gamma}{di}\right).\end{aligned}$$

The value of $\cos\gamma$ gives

$$\left(\frac{d.\cos\gamma}{d\theta}\right) = \sin(\theta_1 - \theta)\sin A \sin i$$

$$\left(\frac{d.\cos\gamma}{di}\right) = \cos(\theta_1 - \theta)\sin A \cos i - \cos A \sin i.$$

The other equations of the triangle enable us to change these values to

$$\left(\frac{d.\cos\gamma}{d\theta}\right) = \sin\gamma \sin\psi \sin i$$

$$\left(\frac{d.\cos\gamma}{di}\right) = \sin\gamma \cos\psi.$$

We have therefore finally

$$\left.\begin{aligned} \frac{di}{dt} &= -k'.\cos\gamma\sin\gamma\sin\psi \\ \frac{d\theta}{dt} &= -k\cos i + k'.\frac{\cos\gamma\sin\gamma\cos\psi}{\sin i}, \end{aligned}\right\} \quad (5)$$

where k and k' have the values,

$$k = \frac{3S.a^3}{4a_0^3(1-e_0^2)^{\frac{3}{2}}}.n: \qquad k' = \frac{(\rho - \frac{1}{2}\varphi)}{a^2}.n.$$

These are the formulæ given by Laplace, Mec. Cel., Tome IV, page 182. The quantity k can be computed accurately for all of the principal planets; but k' is not so well known since it depends on the ellipticity of the figure of the planet and on its law of density. If the planet has other satellites the secular perturbations of these should be comprised in the value of k', and in the case of Saturn k' would include also the secular action of the Ring.

The Italian astronomer Plana criticised equations (5), and considered them erroneous; but in the *Conn. des Tems* for 1829, p. 245, Laplace has verified his first proof of them, and has given also the simple derivation from the formulæ of Lagrange which has been followed above.

(3). In his investigation of the motion of the orbit of Japetus Laplace has introduced an auxiliary fixed plane passing through the line of nodes and placed between the equator of the planet and its orbit, the angle which this plane makes with the equator of the planet depending on the ratio of k to k'.

M. Tisserand has employed a property of the perturbative function which leads very simply to a knowledge of the curve described on the heavens by the pole of the orbit of the satellite. If ε be the mean longitude at the epoch, and π be the longitude of the inferior apsis, the complete differential of R is

$$dR = \frac{dR}{da}.da + \frac{dR}{d\varepsilon}.d\varepsilon + \frac{dR}{de}.de + \frac{dR}{d\pi}.d\pi + \frac{dR}{di}.di + \frac{dR}{d\theta}.d\theta,$$

but for the secular perturbations we have

$$da = 0, \qquad \text{and } \frac{dR}{d\varepsilon} = 0,$$

and hence

$$dR = \frac{dR}{de}.de + \frac{dR}{d\pi}.d\pi + \frac{dR}{di}.di + \frac{dR}{d\theta}.d\theta.$$

Now we have by the theory of Lagrange,

$$\frac{dR}{de} = (e, \pi).\frac{d\pi}{dt} + (e, \theta).\frac{d\theta}{dt}$$

$$\frac{dR}{d\pi} = (\pi, e).\frac{de}{dt}$$

$$\frac{dR}{di} = (i, \theta).\frac{d\theta}{dt}$$

$$\frac{dR}{d\theta} = (\theta, i).\frac{di}{dt} + (\theta, e).\frac{de}{dt}.$$

If we substitute these values of the partial derivatives of R in the second value of dR and notice the well known relations

$$(e, \pi) = -(\pi, e), \text{ \&c.},$$

we shall have

$$dR = 0,$$

and hence

$$R = \text{constant}.$$

From the value of R that we have found we have,

$$k \sin i^2 + k' \sin \gamma^2 = \text{constant}. \tag{6}$$

If a satellite therefore be acted on by the two disturbing forces that we have considered the pole of its orbit will describe on the heavens a spherical ellipse. This equation shows also that if the disturbing force of the sun be zero, the orbit of the satellite will have a constant inclination to the equator of the planet, and if k' vanish, it will have a constant inclination to the orbit of the planet. The real position of the orbit of the satellite will depend on the ratio of the quantities k and k', and for this ratio we have

$$\frac{k'}{k} = \frac{4}{3}.\frac{a_0^3}{a^3 S}.\frac{1}{a^2}.\left(1-e_0^2\right)^{\frac{3}{2}}.\left(\rho-\frac{\varphi}{2}\right),$$

or, since the mass of the planet is the unit of mass, and

$$1 = a^3n^2, \qquad S = a_0^3 n_0^2;$$

$$\frac{k'}{k} = \frac{4}{3}.\left(\frac{n}{n_0}\right)^2.\frac{1}{a^2}.\left(1-e_0^2\right)^{\frac{3}{2}}.\left(\rho-\frac{\varphi}{2}\right), \tag{7}$$

where the unit of distance is the equatorial radius of the planet. The uncertainty of this ratio lies in our ignorance of the factor $(\rho-\frac{1}{2}\varphi)$. For the

planets Mars, Jupiter and Saturn the quantity φ is known with a tolerable degree of accuracy, since it depends on the time of rotation of the planet about its axis; but the value of ρ must be inferred from the probable conformation of our Earth, or from some other analogy. In the case of the exterior planets of our system the ratio $k' \div k$ is probably large for all the satellites, with one exception, and for this reason these satellites are found nearly in the equators of the planets. This ratio, it will be seen, varies inversely as the fifth power of the mean distance of the satellite, and it is on account of this fact that Japetus, the outer satellite of the Saturnian system, presents an exception to the general rule, its orbit being inclined nearly fourteen degrees to the equator of Saturn, while the next interior satellites, Hyperion and Titan do not depart more than one or two degrees from the equator. In the case of Japetus Laplace found,

$$\frac{k'}{k} = 0.4219,$$

which is too large on account of the uncertain data used in the calculation. M. Tisserand finds from Bessel's value of the mass of the Ring and ellipticity of the planet,

$$\frac{k'}{k} = 0.2570;$$

and this ratio is still uncertain since the mass of the Ring is not well known, and the masses of all the satellites of Saturn are unknown. The better way probably would be to determine k' by observations made at two distant epochs which would enable us to fix the position of the node at those epochs and the values of the inclination, and by comparison of these values we should have k'. In order to distinguish in the case of Saturn the different parts of k' it will be necessary to determine accurately the motions of the interior satellites, since from these motions the mass of the Ring and the effect of the ellipticity of the planet can be found.

For the application of equations (5) to Japetus we need the values of the inclination and longitude of the node of the orbits of Saturn and Japetus and of the equator of Saturn on the ecliptic. These values are for 1880.0

	N.		J.	
Saturn	112°	36′.5	2°	29′.6
Japetus	142	45 .0	18	31 .5
Equator of Saturn	167	55 .2	28	10 .3

The solutions of the three spherical triangles formed by the nodes of these great circles give

$$\begin{aligned} i &= 16^\circ \quad 25'.0 \\ \gamma &= 13 \quad\ \ 41\,.0 \\ \psi &= 53 \quad\ \ 39\,.0 \\ A &= 26 \quad\ \ 49\,.6. \end{aligned}$$

The periodic times of Saturn and Japetus are

$$\begin{aligned} \tau_0 &= 10759.2198 \text{ days} \\ \tau &= \quad\ 79.32936 \text{ ``} \end{aligned}$$

The value of k may be written,

$$k = \frac{3}{4} \cdot \frac{\tau^2}{\tau_0^2} \cdot \frac{n}{(1-e_0^2)^{\frac{3}{2}}}.$$

For Saturn $e_0 = 0.0559956$; and if we take the Julian year of 365.25 days for the unit of time we have for Japetus $n = 5967071''.2$. The value of k is

$$k = 0.0000409651\,.\,n.$$

Equations (5) give for the anuual variations of i and θ,

$$\varDelta i = -\,45''.250.\frac{k'}{k}$$

$$\varDelta\theta = -\,234''.476+117''.827\,.\frac{k'}{k}.$$

With Tisserand's value of $k' \div k$ the annual motion of the node on the orbit of Saturn will be $-204''.194$.

For the outer satellite of Mars the values of the nodes and inclinations on the ecliptic for 1880.0 are:

	N.		J.	
Mars	48°	37′.9	1°	51′.0
Deimos	85	37 .4	25	47 .2
Equator of Mars	79	39 .0	28	50 .0

With regard to these quantities it must be noticed that the position of the equator of Mars is uncertain; but from the data given we find,

$$\begin{aligned} i &= 24^\circ \quad 20'.0 \\ \gamma &= \ \ 4 \quad\ \ \ 5\,.6 \\ \psi &= 312 \quad 38\,.5 \\ A &= \ 27 \quad 15\,.9 \end{aligned}$$

The periodic times of Mars and Deimos are

$$\begin{aligned} \tau_0 &= 686.97965 \quad \text{days} \\ \tau &= \quad 1.2624350 \quad \text{``} \end{aligned}$$

and in the orbit of Mars $e_0 = 0.09328975$; and for the satellite

$$\log n = 8.5739862.$$

The value of k is

$$k = 0.0000025661722 . n.$$

and equations (5) give for the annual motions of i and θ on the orbit of Mars:

$$\Delta i = + 50''.396 . \frac{k'}{k}$$

$$\Delta\theta = - 876''.735 + 112''.630 . \frac{k'}{k}.$$

The node would have therefore a retrograde motion by which it would complete a revolution on the heavens in 1478 years were it not for the influence of the second term. In the case of Mars the value of the ratio $k' \div k$ is unknown, but probably it is large enough to destroy the first term in the value of $\Delta\theta$.

(4). To find the equation of a spherical ellipse, let 2α be the part of a great circle forming the major axis of the ellipse, and let 2ε be the distance between the foci. Denote by ρ and ρ' the arcs drawn from a point on the ellipse to the foci, and we shall have the condition,

$$\rho + \rho' = 2\alpha.$$

If we take the centre of the ellipse as the origin of the polar spherical coordinates S and V, S being the radius vector and V its angle with the major axis, we have,

$$\cos \rho = \cos S \cos \varepsilon - \sin S \sin \varepsilon \cos V$$
$$\cos \rho' = \cos S \cos \varepsilon + \sin S \sin \varepsilon \cos V.$$

Adding and subtracting these equations, and reducing, we find

$$\cos \tfrac{1}{2}(\rho' - \rho) = \frac{\cos S \cos \varepsilon}{\cos \alpha},$$

$$\sin \tfrac{1}{2}(\rho' - \rho) = \frac{- \sin S \sin \varepsilon \cos V}{\sin \alpha};$$

and squaring and adding these we have for the equation of the ellipse,

$$1 = \frac{\cos \varepsilon^2}{\cos \alpha^2} . \cos S^2 + \frac{\sin \varepsilon^2}{\sin \alpha^2} . \sin S^2 \cos V^2. \tag{8}$$

The form of our perturbative function is the same as that of equation (8) since it may be written,

$$R = k \cos i^2 + k' \cos \gamma^2 = c.$$

The spherical ellipse will be symmetrical with respect to the great circle joining the pole of the orbit of the planet to the pole of its equator, and therefore the centre of the ellipse will be on this circle and between these poles. To find this centre M. Tisserand proceeds as follows; let η and η'

be the distances of the centre from the pole of the orbit of the planet and from the pole of the equator. Then

$$\eta + \eta' = A.$$

We shall have from the spherical triangles

$$\cos i = \cos \eta' \cos S - \sin \eta' \sin S \cos V$$
$$\cos \gamma = \cos \eta \cos S + \sin \eta \sin S \cos V.$$

The equation of the ellipse will be

$$k.(\cos \eta' \cos S - \sin \eta' \sin S \cos V)^2 + k'.(\cos \eta \cos S + \sin \eta \sin S \cos V)^2 = c.$$

By equation (8) the coefficient of $\sin S \cos S \cos V$ must be zero. Hence

$$-2k.\cos \eta' \sin \eta' + 2k'.\cos \eta \sin \eta = 0,$$

or

$$k.\sin 2(A - \eta) = k'.\sin 2\eta,$$

and

$$\text{tang}\ 2\eta = \frac{k.\sin 2A}{k' + k \cos 2A}.$$

The angle η is the same as the angle θ used by Laplace, Méc. Céleste Tome IV, p. 178. For two satellites of the same planet the ratio $k' \div k$ will vary inversely as the fifth power of their mean distances. If we assume for Japetus this ratio to be

$$\frac{k'}{k} = 0.2570,$$

its value for Rhea, the fifth satellite of Saturn, will be

$$\frac{k'}{k} = 0.257 \times \left(\frac{64.54}{9.55}\right)^5$$

Hence $\eta = 22''.9$; or, for an observer on our Earth Rhea will move sensibly in the equator of Saturn. For Titan, the large satellite next beyond Rhea, we have

$$\frac{k'}{k} = 0.257 \times \left(\frac{64.54}{22.145}\right)^5.$$

Hence $\eta = 1520''.5$. This is smaller than the value given by observation, and indicates that the assumed ratio

$$\frac{k'}{k} = 0.2570$$

is too great.